Assignment $\frac{25}{2}$.

This homework is due *Friday* Dec 5.

This assignment is worth one half of normal homework in terms of course grade, and is not included in denominator of your course grade. That is, it's a freebie.

There are total 17 points in this assignment. 15 points is considered 100%. If you go over 15 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 7.1 in Bartle–Sherbert.

(1) [1pt] (Part of 7.1.1) If $I = [0, 4]$, calculate the norms of the following partitions:
  (a) $\mathcal{P}_1 = (0, 1, 2, 4)$,
  (b) $\mathcal{P}_2 = (0, 2, 3, 4)$,
  (c) $\mathcal{P}_3 = (0, 1, 1.5, 2, 3.4, 4)$.

(2) (Part of 7.1.2) If $f(x) = x^2$ for $x \in [0, 4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in the previous problem, and the tags are selected as indicated.
  (a) [1pt] $\mathcal{P}_1$ with the tags at the left endpoints of the subintervals.
  (b) [1pt] $\mathcal{P}_2$ with the tags at the right endpoints of the subintervals.

(3) [2pt] (7.1.8) If $f \in \mathcal{R}[a, b]$ and $|f(x)| \le M$ for all $x \in [a, b]$, show that

$$\left| \int_a^b f \right| \le M(b - a).$$

(4) (a) [2pt] (7.1.9) If $f \in \mathcal{R}[a, b]$ and if $(\dot{\mathcal{P}}_n)$ is any sequence of tagged partitions of $[a, b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$, prove that $\int_a^b f = \lim_{n\to\infty} S(f; \dot{\mathcal{P}}_n)$.
  (b) [2pt] (7.1.10) Let $g(x) = 0$ if $x \in [0, 1]$ is rational and $g(x) = 1/x$ if $x \in [0, 1]$ is irrational. Prove that $g \notin \mathcal{R}[0, 1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a, b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} S(g; \dot{\mathcal{P}}_n)$ exists.
  (c) [2pt] Use (a) to compute $\int_a^b x$. (*Hint:* In an arbitrary partition $\mathcal{P}$, pick tags to be $t_i = (x_{i-1} + x_i)/2$. Compute the corresponding Riemann sums.)

— *see next page* —

(5) (a) [2pt] Let $c \in [a, b]$. Let $f : [a, b] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 \text{ if } x \neq c, \\ 1 \text{ if } x = c. \end{cases}$$

Prove that $f \in \mathcal{R}[a, b]$ and $\int_a^b f = 0$.

(b) [2pt] Let $g : [a, b] \to \mathbb{R}$ be 0 on $[a, b]$, except for finitely many points $c_1, \ldots, c_k \in [a, b]$ where $g(c_k) \neq 0$. Prove that $g \in \mathcal{R}[a, b]$ and $\int_a^b g = 0$. (*Hint:* Express g as a linear combination of functions like in (a).)

(c) [2pt] Suppose $h \in \mathcal{R}[a, b]$. Let $\hat{h}$ be obtained h by changing value of h at finitely many points $c_1, \ldots, c_k \in [a, b]$. Prove that $\hat{h} \in \mathcal{R}[a, b]$ and $\int_a^b h = \int_a^b \hat{h}$. (*Hint:* Represent $\hat{h} = h + g$ where g satisfies conditions of (b).)